

# Large Overlaid Cognitive Radio Networks: From Throughput Scaling to Asymptotic Multiplexing Gain

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**Abstract**—We study the asymptotic performance of two multi-hop overlaid ad-hoc networks that utilize the same temporal, spectral, and spatial resources based on random access schemes. The primary network consists of Poisson distributed legacy users with density  $\lambda^{(p)}$  and the secondary network consists of Poisson distributed cognitive radio users with density  $\lambda^{(s)} = (\lambda^{(p)})^\beta$  ( $\beta > 0$ ,  $\beta \neq 1$ ) that utilize the spectrum opportunistically. Both networks are decentralized and deploy ALOHA protocols where the secondary nodes are equipped with range-limited *perfect* spectrum sensors to monitor and protect primary transmissions. We study the problem in two distinct regimes, namely  $\beta > 1$  and  $0 < \beta < 1$ . We show that in both cases, the two networks can achieve their corresponding stand-alone throughput scaling even without secondary spectrum sensing (i.e., sensing range set to zero), which implies the need for a more comprehensive performance metric than just throughput scaling to evaluate the influence of the overlaid interactions. We thus introduce a new criterion, termed the *asymptotic multiplexing gain*, which captures the effect of spectrum sensing and inter-network interferences. Furthermore, based on this metric we demonstrate that spectrum sensing can substantially improve the network performance when  $\beta > 1$ . On the contrary, spectrum sensing turns out to be unnecessary when  $\beta < 1$ .

**Index Terms**—Cognitive Radios, Spectrum Sensing, Geometric Routing Schemes, Asymptotic Multiplexing Gain.

## I. INTRODUCTION

GUPTA and Kumar [1] introduced a random network model for studying the throughput of large-scale static wireless networks where the network consists of  $n$  nodes that are independently and uniformly distributed on a unit-area disk. Each node in the network can act as a source, a relay, or a destination, and each source node has a random destination in the network. The nodes have a common transmission range and each transmits to its one-hop neighbors. They showed that a centralized time-slotted multi-hop transmission scheme can achieve a sum throughput scaling of  $\Theta(\sqrt{n/\log(n)})^1$ . Following [1], there have been a vast literature, e.g., [2]–[5], studying the asymptotic performance of single large-scale network, all based on traditional static spectrum allocation schemes.

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<sup>1</sup> $f(n) = o(g(n))$  means that  $\lim_{n \rightarrow \infty} f(n)/g(n) \rightarrow 0$  as  $n \rightarrow \infty$ ,  $f(n) = O(g(n))$  means that there exist positive constants  $c_1$  and  $M$  such that  $f(n)/g(n) \leq c_1$  whenever  $n \geq M$ ,  $f(n) = \omega(g(n))$  means that  $\lim_{n \rightarrow \infty} f(n)/g(n) \rightarrow \infty$  as  $n \rightarrow \infty$ ,  $f(n) = \Theta(g(n))$  means that both  $f(n) = O(g(n))$  and  $g(n) = O(f(n))$ ,  $f(n) \sim g(n)$  means that  $\lim_{n \rightarrow \infty} f(n)/g(n) \rightarrow 1$  as  $n \rightarrow \infty$ .

The conventional wireless communication systems are not able to cope with the increasing demand for frequency spectrum in the near future. Fortunately, although most portions of the spectrum are crowded with licensed users, they are scarcely used in different locations and at different times [6]. In the seminal work of [7], Mitola proposed *cognitive radio* as a promising solution to utilize the frequency spectrum more efficiently. The underlying idea is to let the unlicensed users (secondary users) make use of the available temporal, spectral, or spatial opportunities over the licensed bands, while protecting the licensed users (primary users) by limiting the interference caused by the secondary users. Therefore, acute secondary interference management schemes are required at secondary users, to maintain certain quality of service (QoS) for the primary network and achieve a reasonable performance for the secondary network in such an overlaid cognitive networks.

In this paper we study the asymptotic performance of multi-hop overlaid networks in which a primary ad-hoc network and a cognitive secondary ad-hoc network coexist over the same spatial, temporal, and spectral dimensions. In order to limit the secondary interference to the primary network, we adopt the *dynamic spectrum access* [8] approach, where secondary users opportunistically explore the white spaces detected using spectrum sensors. In [9], Vu *et al.* considered the throughput scaling law for single-hop overlaid cognitive radio networks, where a linear scaling law is obtained for the secondary network with an outage constraint considered for the primary network. In [10], Jeon *et al.* considered a multi-hop cognitive network coexisting with a primary network and assumed that the secondary nodes know the locations of all primary nodes (both transmitters and receivers). They showed that by defining a preservation region around each primary node and following time-slotted deterministic transmission protocols, both networks can achieve the same throughput scaling law as a stand-alone wireless network, while a vanishing fraction of the secondary nodes may suffer from a finite outage probability (as the number of the nodes tends to infinity). In [11], the authors studied the throughput and throughput-delay tradeoff with the same system model as in [10], except that the secondary users only know the locations of the primary transmitters. By establishing preservation regions around primary transmitters, they showed that both networks could achieve the throughput scaling law derived by Gupta and Kumar in [1] without outage.

In all the previously mentioned papers, centralized deterministic schemes are used to achieve the feasible rates for

both primary and secondary networks. Moreover, the results are only provided for the scenario where the secondary nodes are much more densely distributed than the primary nodes. The desired autonomous features of large wireless systems, make the use of a central authority to coordinate the primary/secondary users less appealing. In addition, in many practical situations, as the secondary users are opportunistic (or sporadic) spectrum utilizers it is more likely that the secondary nodes are less densely distributed. In the literature, the asymptotic performance of traditional single-tier networks with distributed random access schemes have been studied, e.g., [12]–[15]. In [12], the performance of the slotted ALOHA protocol in a multi-hop environment is studied and the optimum transmission radius is derived to maximize the throughput for a random planar network. The spatial capacity of a slotted multi-hop network with capture is studied in [13]. In both of the above papers, the authors showed that the sum throughput of the system scales as  $\Theta(\sqrt{\lambda})$ . In [14], Weber *et al.* derived the transmission capacity of wireless ad-hoc networks, where transmission capacity is defined as the product of the maximum density of successful transmissions multiplied by their data rate, given an outage constraint. Baccelli *et al.* [15] proposed an ALOHA-based protocol for multi-hop wireless networks in which nodes are randomly located in an infinite plane according to a Poisson point process and are mobile according to a waypoint mobility model. They derived the optimum *multiple access probability* that achieves the maximum *mean density of progress*.

In this work we consider decentralized ALOHA-based scheduling schemes for both primary and secondary networks in an overlaid scenario, where secondary users can only make use of localized information obtained via spectrum sensing to control their actions and limit their interferences to primary users. The distributed nature of ad-hoc networks and the passive property of primary receivers lead to uncertainties about the primary system state even with perfect spectrum sensing. As such, we focus on the case where the secondary users are able to *perfectly detect* the primary user signals when the primary transmitters are within a certain range. In particular, we study the asymptotic performance of the two overlaid networks, where we start with the throughput scaling laws, and then introduce a new metric called *asymptotic multiplexing gain* that further quantifies the performance tradeoff between the two networks. We do so under two scenarios: the secondary network is denser vs. sparser than the primary network, and identify their key differences. To the best of our knowledge, this is the first time that the achievable rates for overlaid cognitive networks with random access schemes is studied, where the secondary network could be either denser or sparser than the primary network.

The rest of the paper is organized as follows. Section II introduces the mathematical model, notations, and definitions. In Section III we consider the *spatial throughput* scaling of the single-tier network. Section IV studies the cognitive overlaid scenario and addresses the tradeoff between the primary and secondary networks by introducing the notion of asymptotic multiplexing gain (AMG). In particular, we show that both

networks can achieve their corresponding single-tier throughput scaling regardless of the setting for the spectrum sensing range. However, for the case with a denser secondary network, spectrum sensing can improve the primary network AMG; whereas, for the case with a sparser secondary network, the primary network AMG cannot be enhanced (asymptotically) by spectrum sensing while maintaining a non-trivial throughput for the secondary network. Section V concludes the paper.

## II. SYSTEM MODEL AND DEFINITIONS

Consider a circular area  $A$  in which a network of primary nodes and a network of secondary nodes share the same temporal, spectral, and spatial resources<sup>2</sup>. Both primary and secondary nodes are distributed according to Poisson point processes with densities  $\lambda^{(p)}$  and  $\lambda^{(s)} = (\lambda^{(p)})^\beta$  ( $\beta > 0, \beta \neq 1$ ), respectively. Let  $\phi^{(p)} = \{X_i^{(p)}\}$  and  $\phi^{(s)} = \{X_i^{(s)}\}$  denote the (Cartesian) coordinates of a realization of the primary and secondary nodes. As mentioned earlier, the primary users are the legacy users, and thus have a higher priority to access the spectrum; secondary users can access the spectrum opportunistically (based on the spectrum sensing outcome) as long as they abide by “certain” interference constraints.

Throughout this paper we denote the parameters associated with the primary and the secondary users with superscripts  $(p)$  and  $(s)$ , respectively; e.g.,  $R_I^{(p)}$  denotes the interference range from a primary transmitter to a primary receiver and  $R_I^{(s)}$  denotes the interference range from a secondary transmitter to a secondary receiver. Each primary receiver tries to decode the signal from its intended transmitter located within  $R_r^{(p)}$  radius and is prone to interference from other primary and secondary transmitters within  $R_I^{(p)}$  and  $R_I^{(sp)}$  radii, respectively. Likewise, a secondary receiver tries to decode the signal from its intended transmitter located within  $R_r^{(s)}$  radius and is prone to interference from other secondary and primary transmitters within  $R_I^{(s)}$  and  $R_I^{(ps)}$  radii, respectively. Furthermore, due to certain cognitive features<sup>3</sup>, we assume that the cognitive secondary receivers are more robust against primary interferences than primary receivers, i.e.,  $R_I^{(ps)} \leq R_I^{(p)}$ ; and the primary receivers are more sensitive to the secondary interference than secondary receivers<sup>4</sup>, i.e.,  $R_I^{(sp)} \geq R_I^{(s)}$ . Also it is reasonable to assume that the interference range is no less than the transmission range for both networks, i.e.,  $R_I^{(p)} = \sqrt{1 + l^{(p)}} R_r^{(p)}$  and  $R_I^{(s)} = \sqrt{1 + l^{(s)}} R_r^{(s)}$  for some constants  $l^{(p)}, l^{(s)} \geq 0$ . In addition, secondary nodes are equipped with perfect spectrum sensors that can detect the primary user signals (i.e., the existence of transmitting primary users) within  $R_D$  radius. Fig. 1 encapsulates the relationship among aforementioned ranges.

Let  $|A|$  denote the area of a region  $A$  and  $\bar{E}$  denote the complement of event  $E$ . Let  $B_R(\cdot)$  denote a full disk with radius  $R$  centered at  $(\cdot)$ , which can be either the polar coordinates in the form of  $(r, \varphi)$  or the location of a node  $X$  in

<sup>2</sup>All the results will carry over to any smooth and convex region with some minor considerations.

<sup>3</sup>e.g., acquiring knowledge about primary message and utilizing joint encoding techniques to partially mitigate primary interference.

<sup>4</sup>e.g., due to possible multiuser cooperation among secondary users.

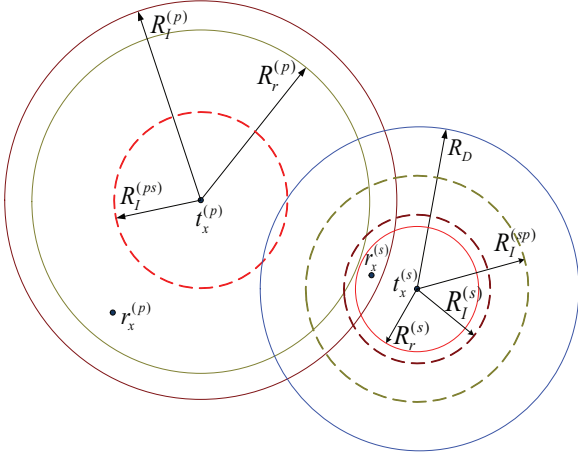


Fig. 1. Primary and secondary network parameters.  $t_x^{(p)}$  and  $r_x^{(p)}$  denote a primary transmitter and receiver, and  $t_x^{(s)}$  and  $r_x^{(s)}$  denote a secondary transmitter and receiver, respectively.

the form of  $(X)$ . Finally, interpret  $B_{R_1}(r_1, \varphi_1) - B_{R_2}(r_2, \varphi_2)$  as the remaining region of a disk with radius  $R_1$  centered at polar coordinates  $(r_1, \varphi_1)$  excluding the overlapping region with another disk with radius  $R_2$  center at  $(r_2, \varphi_2)$ .

For the transmission protocols in both networks, the time axis is slotted and the slot duration is defined as the time required to transmit a packet in the system, where all packets are assumed to be of the same size. In the following, we outline the primary and secondary network protocols, both based on the slotted ALOHA structure.

#### A. Primary Network Protocol

Each primary node picks a destination uniformly at random among all other nodes in the primary network. Communication occurs between a primary source-destination (S-D) pair through a single-hop transmission if they are close enough, or through multi-hop transmissions over intermediate relaying nodes if they are far apart. In this manner, each primary node might act as a source, destination or a relay, and always has a packet to transmit (which is either its own-generated packet or a packet being relayed). We assume that each node has an infinite queue for packets where the first packet in the queue is transmitted with probability  $q^{(p)}$  (the ALOHA parameter). The selection of relaying nodes along the (multi-hop) routing path is governed by a variant of geometric routing schemes, namely the *random 1/2-disk routing scheme*<sup>5</sup> as discussed in Section II-C.

#### B. Secondary Network Protocol

Similar to the primary network, each secondary node picks a destination uniformly at random among all other nodes in the secondary network. Each secondary node has an infinite queue for packets with the first one in the queue transmitted with probability  $q^{(s)}$ , whenever the channel is deemed idle:

<sup>5</sup>We choose the random 1/2-disk routing scheme mainly for tractability and simplicity in mathematical characterization. However, the solution techniques developed in this paper can be used (with some modifications) to study other variants of geographical routing schemes, such as MFR, NFP, DIR, etc.

In particular, each secondary user senses the channel for primary activities prior to a transmission initiation and commences the transmission of the first packet in the queue with probability  $q^{(p)}$  whenever there are no primary transmitters detected within  $R_D$  radius. Setting  $R_D = 0$  implies that secondary nodes always initiate transmissions with probability  $q^{(s)}$  regardless of the primary channel occupancy status. The secondary network utilizes a similar routing scheme to that in the primary network.

#### C. Random 1/2-Disk Routing Scheme

Since both primary and secondary networks utilize the same routing scheme, in this section, we introduce our routing scheme for a generic wireless ad-hoc network. Throughout the paper, we assume that both primary and secondary networks possess the following property: each network node has at least one relaying node in every direction; This is a sufficient condition for the existence of routing paths (with finite lengths) between any arbitrary source-destination pair in the network and can be guaranteed asymptotically almost surely if  $R_r = K\sqrt{\log \lambda/\lambda}$  for a large enough constant  $K$  (c.f. [17], Theorem 1).

Consider an arbitrary packet  $b$  with the source-destination pair that is  $h$ -distance apart. We set the destination node at the origin and assume that the routing path starts from the source node at  $X_0 = (-h, 0)$ , where  $X_n$  is the (Cartesian) coordinate of the  $n^{\text{th}}$  relay node along the routing path and  $r_n := \|X_n\|$  is the (Euclidean) distance of the  $n^{\text{th}}$  relay node from the destination.

More specifically, the routing path starts at the source node  $X_0 = (-h, 0)$  with its transmission 1/2-disk  $D_0^b$  that is a 1/2-disk with radius  $R_r$  centered at  $X_0$  and oriented towards the destination at  $(0, 0)$ . The next relay  $X_1$  is selected at random from nodes contained in  $D_0^b$ . This induces a new 1/2-disk  $D_1^b$ , centered at  $X_1$  and oriented towards the destination. Relay  $X_2$  is selected randomly among the nodes in  $D_1^b$ , and the process continues in the same manner until the destination is within the transmission range. We claim that the routing path has converged (or is established) whenever it enters the transmission/reception range of the final destination, i.e.,  $r_\tau \leq R$ , for some  $\tau \in \{1, 2, \dots\}$ . In Fig. 2, we illustrate the progress of routing towards the destination. Define the *progress* at the  $n^{\text{th}}$  hop of the routing path as  $Y_{n+1} := \|X_n\| - \|X_{n+1}\| = r_n - r_{n+1}$ ,

#### D. Spatial Throughput

In this paper we adopt a notion of throughput similar to *mean spatial density of progress* in [15].

**Definition 1.** We define the *spatial throughput of the network as the mean total progress of all successfully transmitted packets in the whole network over a single hop. More specifically, let  $b_{[i]}$  be the packet at the head of node  $X_i$ 's queue,  $Y_{X_i}^{b_{[i]}}$  be the progress of packet  $b_{[i]}$  at node  $X_i$  and  $\Lambda_{X_i}^{b_{[i]}}$  be the event of successful transmission of packet  $b_{[i]}$  at node  $X_i$ . Then the*



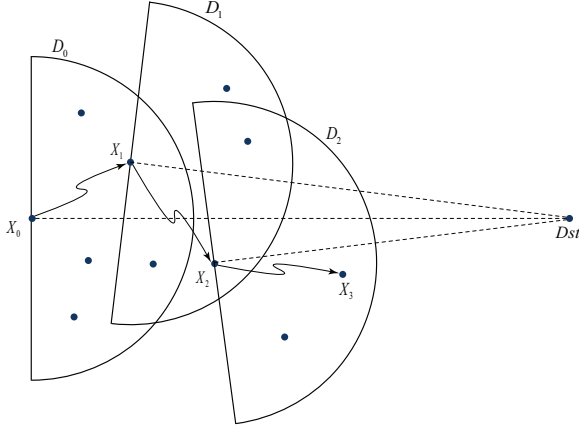


Fig. 2. Evolution of the random 1/2-disk routing path.

*spatial throughput of the network is defined as*

$$C = E \left( \sum_{X_i \in \phi} Y_{X_i}^{b_{[i]}} \mathbf{1}_{\Lambda_{X_i}^{b_{[i]}}} \right), \quad (1)$$

where  $\mathbf{1}$  is the indicator function and the expectation is taken over all realization of the network nodes, source-destination pairs and the routing paths.

The key different between our notion of throughput and the mean spatial density of progress lies in the definition of the progress, where in [15] the progress is defined to be the decrement in the distance of the packet's position projected on the line connecting the transmitting node and the destination, whereas in this paper we define the progress to be the decrement in the radial distance of a packet to its destination, as shown in Fig. 3. In order to highlight the difference between these two definitions, consider the following exaggerated example.

Assume a routing scheme that always chooses a node in the upper/lower corner of the transmission 1/2-disk as the next relay (e.g.,  $\tilde{X}$  in Fig. 3). Based on this scheme, at each hop, the packet gets farther away from the destination and should never reach the destination; this is an intuitive result that our definition of progress complies with, however, according to the projected distance progress definition, at each hop, the packet has made a positive drift towards the destination and should eventually reach the destination. In the next section, we determine the spatial throughput of the stand-alone primary and secondary networks and point out some of the interpretations for this metric.

### III. SINGLE NETWORK THROUGHPUT SCALINGS

In this section we consider the spatial throughput of a single-tier network when no other networks are overlaid. This serves as a performance benchmark for the overlaid case discussed in the next section. The following lemma provides us with an equivalent definition and a method of computing the spatial throughput for our system.

**Lemma 1** (Separation Principle). *The spatial throughput of the single-tier version of the wireless ad-hoc network defined*

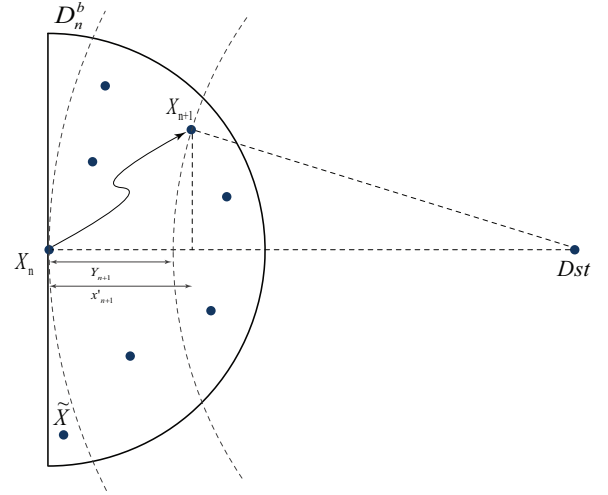


Fig. 3. Progress of the packet at the  $n^{\text{th}}$  hop: a) decrement in the radial distance of a packet to its destination ( $Y_{n+1}$ ) b) decrement in the distance of the projection of the packets position on the line connecting the transmitting node and the destination ( $x'_{n+1}$ )

in Section II equals the product of the expected number of simultaneously successful transmissions in the whole network by the average progress of a typical packet over a single-hop transmission. More formally, let  $\Lambda_X$  be the event of a successful transmission and  $Y_X$  be the progress of a typical packet  $b$  at a typical node  $X$  in the network. Then the spatial throughput of the network can be obtained by

$$C = \lambda |A| \Pr(\Lambda_X) E(Y_X), \quad (2)$$

where the expectation is taken over all realization of the network nodes, source-destination pairs and the routing paths

*Proof:* Let  $b_{[i]}$  be the packet at the head of node  $X_i$ 's queue at an arbitrary time slot. Also, let  $\Lambda_{X_i}^{b_{[i]}}$  be the event of successful transmission of packet  $b_{[i]}$  and  $Y_{X_i}^{b_{[i]}}$  be the progress of  $b_{[i]}$  at node  $X_i$  (given its transmission is successful). (Note that  $b_{[i]}$  and  $Y_{X_i}^{b_{[i]}}$  are random variables dependent on the specific realization of the network nodes, source-destination assignments, and routing path establishment by the random 1/2-disk routing scheme.) The packet  $b_{[i]}$  is successfully transmitted/relayed from  $X_i$  to the next relay, if

- I) Node  $X_i$  initiates a transmission according to the ALOHA protocol (denoted by event  $\Lambda_{1,X_i}^{b_{[i]}}$ ).
- II) For any node  $X_j$  in the transmission 1/2-disk  $D_{X_i}^{b_{[i]}}$  of  $X_i$  that is selected as the next relay for  $b_{[i]}$  (according to the random 1/2-disk routing scheme), we have that node  $X_j$  does not initiate a transmission together with all the other nodes contained in its interference range  $B_{R_I}(X_j)$  (except  $X_i$ ) (denoted by event  $\Lambda_{2,X_j}^{b_{[i]}}$ ).

Note that since  $R_I > R_r$ ,  $\Lambda_{2,X_j}^{b_{[i]}}$  also implies that no two nodes will transmit packets to node  $X_j$  at the same time. Hence,  $\Lambda_{X_i}^{b_{[i]}}$  only depends on the MAC decisions of  $X_i$  and the nodes that are contained in the interference range of  $X_j$ . All these nodes initiate transmissions independent of each other and independent of all previous transmission attempts.

Together with the fact that all network nodes always have a packet to transmit, we can conclude that  $\Pr(\Lambda_{X_i}^{b[i]})$  only depends on the number of nodes contained in the interference range of the next relay node. Hence, due to the homogeneity of underlying Poisson distribution of the network nodes,  $\Pr(\Lambda_{X_i}^{b[i]})$  is only a function of the area of the  $B_{R_I}(X_j)$ , and not the specific realization of  $X_j$ . In other words,  $\{\Lambda_{X_i}^{b[i]}\}$  are identically distributed (but possibly correlated) random variables, and are independent of  $\|X_i\|$ ,  $\|X_j\|$ , and consequently  $Y_{X_i}^{b[i]}$ , which yields (2) directly from (1). ■

As a consequence of Lemma 1, we can derive the spatial throughput of the network by separately determining the probability of a successful one-hop transmission and the average progress for a typical packet  $b$  at a typical node  $X$ . Based on the proof of Lemma 1 we get that

$$\begin{aligned} \Pr(\Lambda_X) &= \mathbb{E} \left( \sum_{X_i \in D_X^b} \frac{q(1-q)^{n_{X_i}} \mathbf{1}_{n_{X_i} > 0}}{\tilde{n}_X} \right) \\ &= qe^{-\lambda q |B_{R_I}(X_i)|} \left( 1 - e^{-\lambda(1-q) |B_{R_I}(X_i)|} \right) \\ &= qe^{-\lambda q \pi R_I^2} \left( 1 - e^{-\lambda(1-q) \pi R_I^2} \right) \end{aligned} \quad (3)$$

where  $\tilde{n}_X \sim \text{Pois}(\lambda |D_X^b|)$  is the number of nodes in  $D_X^b$  and  $n_{X_i} \sim \text{Pois}(\lambda |B_{R_I}(X_i)|)$  is the number of nodes in the interference range of  $X_i$  (excluding  $X_i$  itself). Observe that based on (3),  $q = \frac{1}{\lambda \pi R_I^2}$  maximizes the probability of successful transmission (when  $\lambda$  is large) and  $q = O(1/\log(\lambda))$  is a *necessary* condition for  $\Pr(\Lambda_X)$  to be nontrivial asymptotically.

In order to derive the average packet progress we need some more nomenclature. Consider a packet  $b$ . To simply the notation we drop the superscripts associated with this packet. According to [17] (c.f., Proposition 1), we can (approximately) model the distance  $\{r_n\}$  of packet  $b$  to its destination as a Markov process solely characterized by its progress  $\{Y_n\}$  at hop  $n$ . Let  $\{X_n\}$  be the set of nodes that  $b$  hops over, and let  $(x'_{n+1}, y'_{n+1})$  be the projection of  $X_{n+1} - X_n$  onto the *local* Cartesian coordinates with node  $X_n$  as the origin and the  $x$ -axis pointing from  $X_n$  to the destination node as shown in Fig. 4. Hence,

$$r_{n+1} = \sqrt{(r_n - x'_{n+1})^2 + y'^2_{n+1}}, \quad (4)$$

According to [17] (Proposition 1),  $X_{n+1}$  is uniformly distributed on  $D_n$  for large enough  $\lambda$ ; hence  $\{(x'_n, y'_n)\}$  is an i.i.d. sequence of random variables with  $0 \leq x'_n \leq R_r$  and  $-R_r \leq y'_n \leq R_r$  for all  $n$ , whenever  $\lambda$  is large enough.

Define  $\nu_r^{(h)} := \inf\{n : r_n \leq r, r_0 = h\}$ ,  $R_r \leq r \leq h$ , to be the index of the last relay node closer than  $r$  to the destination when the source and destination nodes are  $h$ -distance apart. Hence,  $\nu_{R_r}^{(h)} + 1$  represents the length of the routing path. In [17] we prove that under certain conditions for  $R_r$ ,  $\nu_r^{(h)}$  is finite asymptotically almost surely. Note that  $\nu_r^{(h)}$  is a stopping time [16] and

$$r - R_r \leq r_{\nu_r^{(h)}} \leq r.$$

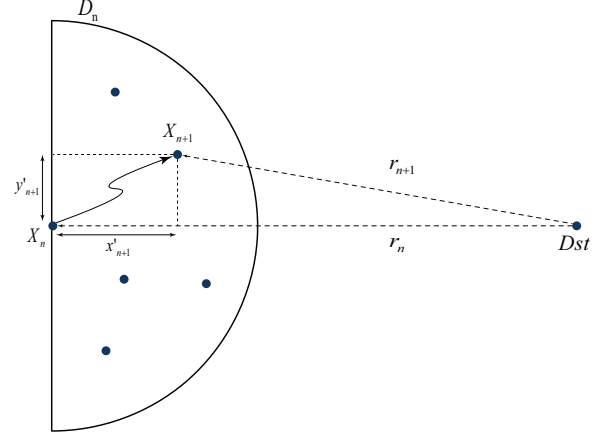


Fig. 4. Distance between the next relay and the current node projected onto the local coordinates at the current node.

Furthermore, let  $g(r, x', y') := \sqrt{(r - x')^2 + y'^2} - r$ . Observe that  $g$  is a nonincreasing function over  $r > R_r$ , for fixed  $(x', y')$ , and  $-g(r_n, x'_{n+1}, y'_{n+1}) = Y_{n+1}$  (the progress at the  $n^{\text{th}}$  relay). Thus, for  $n < \nu_r^{(h)}$ , we have  $r_n > r$  and

$$\begin{aligned} -x'_{n+1} &\leq r_{n+1} - r_n = g(r_n, x'_{n+1}, y'_{n+1}) \\ &\leq g(r, x'_{n+1}, y'_{n+1}) \leq -x'_{n+1} + \frac{R_r^2}{2r}. \end{aligned} \quad (5)$$

Hence, for a source-destination pair that is  $h$ -distance apart ( $r_0 = h$ ), we have

$$r - R_r \leq r_{\nu_r^{(h)}} \leq h + \sum_{n=1}^{\nu_r^{(h)}} g(r, x'_n, y'_n), \quad (6a)$$

$$h + \sum_{n=1}^{\nu_r^{(h)}} (-x'_n) \leq r_{\nu_r^{(h)}} \leq r. \quad (6b)$$

which together with (5) yields

$$\begin{aligned} \mathbb{E} \left( \frac{h - r}{\nu_r^{(h)}} \right) &\leq \mathbb{E}(Y_X) = \mathbb{E}(Y_X^b) \\ &= \mathbb{E} \left( \frac{1}{\nu_r^{(h)}} \sum_{n=1}^{\nu_r^{(h)}} Y_n \right) \\ &= \mathbb{E} \left( \frac{1}{\nu_r^{(h)}} \sum_{n=1}^{\nu_r^{(h)}} -g(r_{n-1}, x'_n, y'_n) \right) \\ &\leq \mathbb{E} \left( \frac{h - r + R_r}{\nu_r^{(h)}} \right) + \frac{R_r^2}{2r}. \end{aligned}$$

where the expectation is taken over all network, S-D assignments, and routing path realizations. Now let  $S_m := \sum_{n=1}^m x'_n$  with  $S_0 = 0$ , and  $\eta(u) := \mathbb{E}(e^{u x'_n})$ . We know that  $e^{u S_m - m \log(\eta(u))}$  is a positive martingale, with value 1 at  $m = 0$ , [16]. Hence we have (recalling (6b))

$$\mathbb{E} \left( e^{u(h-r) - \nu_r^{(h)} \log(\eta(u))} \right) \leq \mathbb{E} \left( e^{u S_{\nu_r^{(h)}} - \nu_r^{(h)} \log(\eta(u))} \right) \leq 1.$$

This implies

$$\mathbb{E} \left( e^{-\nu_r^{(h)} \log(\eta(u))} \right) \leq e^{-u(h-r)}.$$

Using Jensen's inequality and monotone convergence theorem [16], it is easy to show that

$$\begin{aligned} \frac{1}{\mathbb{E}(\nu_{R_r}^{(h)})} &\leq \mathbb{E} \left( \frac{1}{\nu_{R_r}^{(h)}} \middle| h \right) = \int_0^\infty e^{-u\nu_{R_r}^{(h)}} du \\ &\leq \int_0^\infty e^{-u(h-r)} d(\log(\eta(u))) \\ &\leq \int_0^\infty \mathbb{E}(x'_n) e^{-u(h-r-R_r)} du \\ &= \frac{\mathbb{E}(x'_n)}{h-r-R_r}. \end{aligned}$$

Finally, choosing  $r = R_r \left(1 + \sqrt{\frac{h}{R_r}}\right)$ , we can determine the average progress of a typical packet at a typical node  $X$  by

$$\mathbb{E}(Y_X) = \frac{4R_r}{3\pi} + O(R_r^{3/2}) \sim \frac{4R_r}{3\pi}, \quad (7)$$

where we have used the facts that  $\mathbb{E}(\nu_{R_r}^{(h)} | h) \sim \frac{h}{\mathbb{E}(x'_n)}$  and  $\mathbb{E}(x'_n) = \frac{4R_r}{3\pi}$ , c.f. [17]. Combining (3) and (7) we obtain the spatial throughput of the single-tier network as

$$\begin{aligned} C &\sim \lambda |A| \Pr(\Lambda_X) \mathbb{E} \left( \frac{h}{\nu_{R_r}^{(h)}} \right) \\ &= \frac{4|A|}{3\pi} \lambda q R_r e^{-\lambda q \pi R_r^2} \left( 1 - e^{-\lambda(1-q)\pi R_r^2} \right) \\ &= \Theta \left( \sqrt{\frac{\lambda}{\log(\lambda)}} \right), \end{aligned} \quad (8)$$

when  $q = O(1/\log(\lambda))$  and  $R_r = O(\sqrt{\log(\lambda)/\lambda})$ .

**Remark 1.** Observe that if the network is stable, the spatial throughput of the network equals the expected number of packet-meters that the network delivers to the destinations at each time slot, which is equivalent to the transport capacity defined in [1]. The network is stable if the rate at which new packets are generated is equal to the rate at which packets are delivered to their respective destinations. In other words, the queue length of all network nodes is almost surely finite and packets are not being stored in some nodes in the network. Intuitively, when the network is stable, there are  $\lambda|A|\Pr(\Lambda_X)$  successful one-hop transmissions occurring in the whole network in each time slot, however, due to relaying, only  $\mathbb{E}(h/\nu_{R_r}^{(h)})$  of these successful transmissions (on average) contribute to the throughput<sup>6</sup> and the rest are only the retransmissions of the already generated packets.

We will later show (in Section IV) that even when we have two networks sharing the same resources and the secondary network accesses the spectrum without sensing (as if the primary tier is not present), both networks can still achieve

the above throughput scalings, with respective parameters. This suggests that throughput scaling alone is not adequate to evaluate the performance of large-scale overlaid networks as it masks the effect of mutual interference between the two networks. Intuitively, when the secondary users try to access the spectrum more aggressively, the primary network throughput should degrade. However, it turns out that the augmented interference from secondary users only causes a constant penalty to the primary throughput in the asymptotic sense such that the scaling law by itself cannot reflect this effect.

To quantify the effect of mutual interference between the two networks, we define a new measure, *asymptotic multiplexing gain* (which should be a function of the spectrum sensing range at the secondary nodes), to characterize the protection vs. competition tradeoff between the two networks.

**Definition 2.** Assume that the throughput  $C(\lambda)$  of a network scales as  $\Theta(f(\lambda))$ ; we define the *Asymptotic Multiplexing Gain (AMG)* of the network as the constant  $\chi$  such that:

$$\chi = \lim_{\lambda \rightarrow \infty} \frac{C(\lambda)}{f(\lambda)}. \quad (9)$$

Note that the exact value of  $\chi$  may not be always computable, but its bounds always are; and based on that we can define a *partial ordering* [18] on the (set of all) network throughputs. Specifically, consider two networks  $A$  and  $B$  with throughputs  $C_A$  and  $C_B$ , and asymptotic multiplexing gains  $x_1 \leq \chi_A \leq x_2$  and  $x'_1 \leq \chi_B \leq x'_2$ . We say  $C_A \preceq C_B$  if and only if  $C_A/C_B = o(1)$ , or  $x_2 \leq x'_1$  when  $C_A/C_B = O(1)$ <sup>7</sup>. From a different perspective, if we plot  $C(n)$  over  $f(n)$  for asymptotically large  $n$ , AMG is nothing but the slope of the throughput scaling curve, hence the connotation ‘‘multiplexing gain’’; and it is intuitive to always desire a large AMG.

Accordingly, we can determine the single-tier network AMGs in the absence of the other one as:

$$\chi = \frac{4|A|e^{-1}}{3\pi}.$$

#### IV. OVERLAID COGNITIVE NETWORK SPATIAL THROUGHPUT

In this section we consider the case where both primary and secondary networks are present in the overlaid fashion under two distinct scenarios: one with the secondary network being denser than the primary network ( $\beta > 1$ ) and the other with the primary network being denser ( $\beta < 1$ ). Note that, according to Lemma 1, the impact of each tier on the other's throughput is materialized in the expected number of successful one-hop transmissions, through the mutual interference perceived by the corresponding receivers. As seen later, what differentiates the above two scenarios is the level of controlled interference each network inflicts on the other one. The throughput analysis of the overlaid networks closely follows the previous section with proper modifications to the calculation of probability of successful transmissions, which now should take the mutual interferences into account.

<sup>6</sup>The temporal analysis of the system is beyond the scope of this paper and will be discussed in a future work.

<sup>7</sup>This definition closely resembles Lexicographic ordering [18].

### A. Throughput Analysis for the Primary Network

In the presence of the secondary network, a packet  $b$  is successful transmitted from a primary node  $X_n^{(p)}$  to the next relay  $X_{n+1}^{(p)}$  if in addition to the event  $\Lambda_{X_n^{(p)}}^b := \Lambda_{1,X_n^{(p)}}^b \cap \Lambda_{2,X_{n+1}^{(p)}}^b$  (c.f., proof of Lemma 1) we have that there are no active secondary transmitters within inter-network interference range  $R_I^{(sp)}$  of  $X_{n+1}^{(p)}$  (denoted by  $\Lambda_{3,X_{n+1}^{(p)}}^b$ ). In particular, we know that each secondary transmitter initiates transmission with probability  $q^{(s)}$  only if it detects the channel as *idle*, i.e., when there are no primary transmitters within  $R_D$  radius of itself. Therefore, if  $X_n^{(p)}$  initiates a transmission all secondary users in  $B_{R_D}(X_n^{(p)})$  would refrain from transmission and we only need to consider the (possible) interference from the secondary nodes in  $B_{R_I^{(sp)}}(X_{n+1}^{(p)}) - B_{R_D}(X_n^{(p)})$ . Subsequently, we can determine the probability of successful transmissions for the primary network as follows. Let us first consider the  $\beta > 1$  scenario. In this case we have  $R_r^{(p)} = \omega(R_r^{(s)})$  and the probability of successful transmission can be obtained as

$$\Pr(\Lambda_{X_n^{(p)}}^b) \Pr(\Lambda_{3,X_{n+1}^{(p)}}^b | \Lambda_{X_n^{(p)}}^b) = \Pr(\Lambda_{X_n^{(p)}}^b) \int_{r=0}^{R_r^{(p)}} e^{-\lambda^{(s)} q^{(s)} (|B_{R_I^{(sp)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(p)})^2},$$

due to the uniform distribution of the primary nodes. Observe that if there is a primary receiver in the interference range of a secondary transmitter, as  $R_D$  increases, it becomes less likely that its corresponding primary transmitter goes undetected. Thus, we see that the probability of successful transmission for a primary user is an increasing function of  $R_D$ . Similarly, setting  $R_D \geq R_I^{(sp)} + R_r^{(p)}$  guarantees zero interference from the secondary network to the primary network. Following the argument in Section III, the primary network spatial throughput can be derived as

$$C_{\{\beta>1\}}^{(p)} = \gamma_{\{\beta>1\}}(R_D, \lambda^{(p)}) C^{(p)},$$

where  $C^{(p)}$  is the single-tier spatial throughput (8) with primary network parameters substituted, and

$$\begin{aligned} \gamma_{\{\beta>1\}}(R_D, \lambda^{(p)}) &= \int_{r=0}^{R_r^{(p)}} e^{-\lambda^{(s)} q^{(s)} (|B_{R_I^{(sp)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(p)})^2} \\ &= \int_{r=0}^{R_D - R_r^{(sp)}} \frac{2rdr}{(R_r^{(p)})^2} \\ &+ \int_{r=R_D - R_r^{(sp)}}^{R_D + R_I^{(sp)}} e^{-\lambda^{(s)} q^{(s)} (|B_{R_I^{(sp)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(p)})^2} \\ &+ \int_{r=R_D + R_I^{(sp)}}^{R_r^{(p)}} e^{-\lambda^{(s)} q^{(s)} (R_I^{(sp)})^2} \frac{2rdr}{(R_r^{(p)})^2}, \end{aligned}$$

which can be bounded as  $0 < \gamma_{\min} \leq \gamma_{\{\beta>1\}}(R_D, \lambda^{(p)}) \leq \gamma_{\max}$ , with  $\gamma_{\min}$  and  $\gamma_{\max}$  given in (10) and (11). Observe that, as in the previous section,  $q^{(p)} = O(1/\log(\lambda^{(p)}))$  is still a necessary condition to ensure an asymptotically nontrivial throughput for the primary network and the primary network

achieves a throughput scaling of  $\Theta(\sqrt{\lambda^{(p)}/\log(\lambda^{(p)})})$  regardless of the value for  $R_D$ . However, note that the primary network AMG  $\chi_{\{\beta>1\}}^{(p)}$  in the presence of secondary network is reduced by

$$\begin{aligned} \frac{\chi_{\{\beta>1\}}^{(p)}}{\chi} &= \lim_{\lambda^{(p)} \rightarrow \infty} \gamma_{\{\beta>1\}}(R_D, \lambda^{(p)}) \\ &= \begin{cases} a_1 & \text{if } R_D = o(R_r^{(p)}) \\ a_1 + (1 - a_1) \alpha^2 & \text{if } R_D = \alpha R_r^{(p)} + o(R_r^{(p)}), \end{cases} \end{aligned}$$

for  $0 < \alpha \leq 1$ ,  $q^{(s)} = O(1/\log(\lambda^{(s)}))$ , and  $a_1 = \exp\left(-\left(R_I^{(sp)}/R_I^{(s)}\right)^2\right)$ .

**Remark 2.** Note that in the presence of the secondary network, the throughput of the primary network is asymptotically degraded by a non-zero constant factor  $\chi_{\{\beta>1\}}^{(p)}/\chi$  that is less than or equal to one, although the scaling law is still the same. However, with the proper choice of the secondary detection range  $R_D$ , primary network AMG loss can be recovered arbitrarily.

Next, let us consider the case with  $\beta < 1$ , where we have much fewer secondary nodes with much larger interference ranges (than primary nodes) in the network, and determine the throughput scaling and the asymptotic multiplexing gain of the primary network. Based on the definition of successful transmissions, we obtain the same expression for the probability of successful transmission and the primary network throughput when  $\beta < 1$ , however, with a different derivation for  $\gamma_{\{\beta<1\}}(R_D, \lambda^{(p)})$  due to  $R_r^{(p)} = o(R_r^{(s)})$ , i.e.,

$$\begin{aligned} \gamma_{\{\beta<1\}}(R_D, \lambda^{(p)}) &= \int_{r=0}^{R_r^{(p)}} e^{-\lambda^{(s)} q^{(s)} (|B_{R_I^{(sp)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(p)})^2} \\ &= \int_{r=0}^{R_I^{(sp)} - R_D} e^{-\lambda^{(s)} q^{(s)} ((R_I^{(sp)})^2 - R_D^2)} \frac{2rdr}{(R_r^{(p)})^2} \\ &+ \int_{r=R_I^{(sp)} - R_D}^{R_r^{(p)}} e^{-\lambda^{(s)} q^{(s)} (|B_{R_I^{(sp)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(p)})^2}. \end{aligned}$$

We will show later (refer to (14)) that when  $\beta < 1$ , the secondary detection range should be no larger than  $kR_r^{(p)}$ , for some constant  $k > 0$ ; otherwise, the secondary throughput diminishes to zero as  $\lambda^{(p)} \rightarrow \infty$ . Hence, we have  $R_I^{(sp)} - R_D > R_r^{(p)}$  for large  $\lambda^{(p)}$  and

$$\gamma_{\{\beta<1\}}(R_D, \lambda^{(p)}) = e^{-\lambda^{(s)} q^{(s)} ((R_I^{(sp)})^2 - R_D^2)}.$$

Consequently, we obtain that the primary network throughput scales as  $\Theta(\sqrt{\lambda^{(p)}/\log(\lambda^{(p)})})$  and the primary network



$$\gamma_{\min} = \begin{cases} \left( \frac{R_D - R_I^{(sp)}}{R_r^{(p)}} \right)^2 + \left( 1 - \left( \frac{R_D - R_I^{(sp)}}{R_r^{(p)}} \right)^2 \right) e^{-\lambda^{(s)} q^{(s)} (R_I^{(sp)})^2} & \text{if } R_D < R_r^{(p)} - R_I^{(sp)} \\ \left( \frac{R_D - R_I^{(sp)}}{R_r^{(p)}} \right)^2 + \left( 1 - \left( \frac{R_D - R_I^{(sp)}}{R_r^{(p)}} \right)^2 \right) e^{-\lambda^{(s)} q^{(s)} (|B_{R_I^{(sp)}}(0,0) - B_{R_D}(R_r^{(p)}, 0)|)} & \text{Otherwise.} \end{cases} \quad (10)$$

$$\gamma_{\max} = \begin{cases} \left( \frac{R_D + R_I^{(sp)}}{R_r^{(p)}} \right)^2 + \left( 1 - \left( \frac{R_D + R_I^{(sp)}}{R_r^{(p)}} \right)^2 \right) e^{-\lambda^{(s)} q^{(s)} (R_I^{(sp)})^2} & \text{if } R_D < R_r^{(p)} - R_I^{(sp)} \\ 1 & \text{Otherwise.} \end{cases} \quad (11)$$

AMG is reduced by

$$\begin{aligned} \frac{\chi_{\{\beta < 1\}}^{(p)}}{\chi} &= \lim_{\lambda^{(p)} \rightarrow \infty} \gamma_{\{\beta < 1\}}(R_D, \lambda^{(p)}) \\ &= \exp \left( - \left( \frac{R_I^{(sp)}}{R_I^{(s)}} \right)^2 \right), \end{aligned} \quad (12)$$

for  $q^{(s)} = O(1/\log(\lambda^{(s)}))$ .

**Remark 3.** This implies that, asymptotically,  $R_D > 0$  does not improve the primary network AMG. Therefore, the secondary users might as well access the channel without performing spectrum sensing (i.e.,  $R_D = 0$ ) when  $\beta < 1$ . Furthermore, the amount of degradation of the primary network AMG is exponentially dependent on the relative sensitivity of primary and secondary receivers to the interference from secondary users, i.e.,  $R_I^{(sp)}/R_I^{(s)}$ .

### B. Throughput Analysis for the Secondary Network

Similar to Section IV-A, we identify the event of successful transmission of a packet  $b$  from a secondary node  $X_n^{(s)}$  to the next relay  $X_{n+1}^{(s)}$  in the presence of the primary network (denoted by  $\Lambda^{(s)}$ ) as the event  $\tilde{\Lambda}_{X_n^{(s)}}^b \cap \Lambda_{3, X_{n+1}^{(s)}}^b$ , where  $\Lambda_{3, X_{n+1}^{(s)}}^b$  denotes the event that there are no primary transmitters within inter-network interference range  $R_I^{(ps)}$  of  $X_{n+1}^{(s)}$ , and  $\tilde{\Lambda}_{X_n^{(s)}}^b := \tilde{\Lambda}_{1, X_n^{(s)}}^b \cap \tilde{\Lambda}_{2, X_{n+1}^{(s)}}^b$  (c.f., proof of Lemma 1), except that unlike the single-tier network, the secondary users initiate transmission with the (reduced) effective access probability  $\tilde{q}^{(s)} = q^{(s)} \exp(-\lambda^{(p)} q^{(p)} \pi R_D^2)$ . (Note that  $\Lambda_{X_n^{(s)}}^b$  denotes the event of successful transmission if all secondary users initiate transmission with probability  $q^{(s)}$  regardless of the spectrum sensing outcome.) The distinctive feature in the secondary network is that the transmission initiation is contingent upon the detection of an idle spectrum. Thus, the larger  $R_D$  is, the smaller the likelihood of secondary transmission initiation is, which likely decreases the secondary network's throughput. Therefore, in view of the discussions in Section IV-A, there exists a tradeoff between primary and secondary network throughputs (or asymptotic multiplexing gains) via the choice of  $R_D$ . However, based on the results in the previous section, we expect drastic differences in the outcome of this tradeoff between two scenarios:  $\beta > 1$  and  $\beta < 1$ . Let us first consider the case of  $\beta > 1$  and evaluate the probability of successful

transmissions for the secondary network as follows<sup>8</sup>.

$$\begin{aligned} \Pr(\Lambda^{(s)} \mathbf{1}_{\beta > 1}) &= \Pr(\tilde{\Lambda}_{X_n^{(s)}} \Lambda_{3, X_{n+1}^{(s)}} \mathbf{1}_{\beta > 1}) \\ &= \Pr(\Lambda_{3, X_{n+1}^{(s)}}) \Pr(\tilde{\Lambda}_{X_n^{(s)}} \mathbf{1}_{\beta > 1} \mid \Lambda_{3, X_{n+1}^{(s)}}), \end{aligned}$$

which can be evaluated in the following two cases:

- 1)  $R_D \leq R_I^{(ps)} - R_I^{(s)} \leq R_I^{(ps)} - R_r^{(s)}$ : In this case the event  $\tilde{\Lambda}_{X_n^{(s)}} \mathbf{1}_{\beta > 1}$  given  $\Lambda_{3, X_{n+1}^{(s)}}$  is independent of the spectrum sensing outcome. In other words, given  $\Lambda_{3, X_{n+1}^{(s)}}$ , secondary users always initiate transmissions with probability  $q^{(s)}$  at an arbitrary time slot. Therefore,

$$\Pr(\Lambda^{(s)} \mathbf{1}_{\beta > 1}) = e^{-\lambda^{(p)} q^{(p)} \pi (R_I^{(ps)})^2} \Pr(\Lambda_{X_n^{(s)}}^{(s)}).$$

- 2)  $R_D > R_I^{(ps)} - R_I^{(s)}$ : (Note that in this case we only consider the range  $R_D \leq R_I^{(sp)} + R_r^{(p)}$ , since  $R_D$  beyond  $R_I^{(sp)} + R_r^{(p)}$  does not improve the primary network AMG, and instead decreases the secondary network AMG). For this case, observe that  $\Lambda_{X_n^{(s)}} \hat{\Lambda}_{3, X_{n+1}^{(s)}} \mathbf{1}_{\beta > 1} \subseteq \Lambda_{X_n^{(s)}} \Lambda_{3, X_{n+1}^{(s)}} \mathbf{1}_{\beta > 1}$  where  $\hat{\Lambda}_{3, X_{n+1}^{(s)}}$  denotes the event that there are no primary transmitters within  $R_D + R_I^{(s)}$  radius of  $X_{n+1}^{(s)}$ ; again, given  $\hat{\Lambda}_{3, X_{n+1}^{(s)}}$ ,  $\tilde{\Lambda}_{X_n^{(s)}} \mathbf{1}_{\beta > 1}$  is independent of the spectrum sensing outcome. Hence, we can lower-bound the probability of successful transmissions as:

$$\begin{aligned} \Pr(\Lambda^{(s)} \mathbf{1}_{\beta > 1}) &\geq \Pr(\hat{\Lambda}_{3, X_{n+1}^{(s)}}) \Pr(\Lambda_{X_n^{(s)}} \mathbf{1}_{\beta > 1} \mid \hat{\Lambda}_{3, X_{n+1}^{(s)}}) \\ &= e^{-\lambda^{(p)} q^{(p)} \pi (R_D + R_I^{(s)})^2} \Pr(\Lambda_{X_n^{(s)}}^{(s)}). \end{aligned}$$

In order to obtain an upper bound for the probability of successful transmissions, consider the following lemmas.

**Lemma 2.** The knowledge regarding the existence of a non-transmitting secondary user decreases the existence likelihood of a certain area void of primary transmitters.

*Proof:* Refer to Appendix A. ■

**Lemma 3.** The probability of event  $E$  happening, i.e., that there exists a certain area void of primary transmitters given both a non-transmitting, as well as a transmitting secondary users are present, is upper-

<sup>8</sup>Henceforth, We drop the superscript  $b$  for brevity.



bounded by the unconditional probability of  $E$  plus the conditional probability of event  $E$  conditioned solely on the existence of the secondary transmitter.

*Proof:* Refer to Appendix B. ■

As a direct consequence of Lemma 3, the following upper bound can be established for the probability of successful transmission:

$$\begin{aligned} \Pr(\Lambda^{(s)} \mathbf{1}_{\beta>1}) &= \Pr(\Lambda_{3,X_{n+1}^{(s)}} \mathbf{1}_{\beta>1} \mid \tilde{\Lambda}_{X_n^{(s)}}) \Pr(\tilde{\Lambda}_{X_n^{(s)}}) \\ &\leq \left[ \Pr(\Lambda_{3,X_{n+1}^{(s)}}) + \Pr(\Lambda_{3,X_{n+1}^{(s)}} \mathbf{1}_{\beta>1} \mid \tilde{\Lambda}_{1,X_n^{(s)}}) \right] \\ &\quad \cdot \Pr(\tilde{\Lambda}_{X_n^{(s)}}), \end{aligned} \quad (13)$$

where,

$$\begin{aligned} \Pr(\Lambda_{3,X_{n+1}^{(s)}} \mathbf{1}_{\beta>1} \mid \tilde{\Lambda}_{1,X_n^{(s)}}) &= \int_{r=0}^{R_r^{(s)}} e^{-\lambda^{(p)} q^{(p)} (|B_{R_I^{(ps)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(s)})^2} \\ &= e^{-\lambda^{(p)} q^{(p)} \pi ((R_I^{(ps)})^2 - R_D^2)^+} \left( \frac{R_I^{(ps)} - R_D}{R_r^{(s)}} \right)^2 \\ &\quad + \int_{r=R_I^{(ps)} - R_D}^{R_r^{(s)}} e^{-\lambda^{(p)} q^{(p)} (|B_{R_I^{(ps)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(s)})^2} \\ &\leq e^{-\lambda^{(p)} q^{(p)} \pi ((R_I^{(ps)})^2 - R_D^2)^+}, \end{aligned}$$

where 1) the first equality is due to the fact that transmission by  $X_n^{(s)}$  implies that there are no primary transmitters present in  $B_{R_D}(X_n^{(s)})$  and we only need to consider the primary transmitters in  $B_{R_I^{(ps)}}(X_{n+1}^{(s)}) - B_{R_D}(X_n^{(s)})$ ,

- 2)  $\Pr(\Lambda_{3,X_{n+1}^{(s)}}) = \exp(-\lambda^{(p)} q^{(p)} \pi (R_I^{(ps)})^2)$ , and  
3)  $(\cdot)^+ := \max\{(\cdot), 0\}$ .

Ultimately, we can derive the secondary network spatial throughput similarly to that in the previous sections as

$$C_{\{\beta>1\}}^{(s)} = \delta_{\{\beta>1\}}(R_D, \lambda^{(p)}) C^{(s)},$$

where  $C^{(s)}$  is the single-tier spatial throughput (8) with secondary network parameters substituted, and  $\delta_{\{\beta>1\}}(R_D, \lambda^{(p)}) = \exp(-\lambda^{(p)} q^{(p)} \pi (R_I^{(ps)})^2)$  if  $R_D \leq R_I^{(ps)} - R_I^{(s)}$  and otherwise bounded as:

$$\begin{aligned} e^{-\lambda^{(p)} q^{(p)} \pi (R_D + R_I^{(s)})^2} &\leq \delta_{\{\beta>1\}}(R_D, \lambda^{(p)}) \leq \frac{\Pr(\tilde{\Lambda}_{X_n^{(s)}})}{\Pr(\Lambda_{X_n^{(s)}})} \\ &\quad \cdot \left[ e^{-\lambda^{(p)} q^{(p)} \pi (R_I^{(ps)})^2} + e^{-\lambda^{(p)} q^{(p)} \pi ((R_I^{(ps)})^2 - R_D^2)^+} \right]. \end{aligned}$$

Together with the fact that  $R_D \leq R_r^{(p)} + R_I^{(s)}$ , we obtain that  $0 < \delta_{\{\beta>1\}}(R_D, \lambda^{(p)}) < 1$ . Therefore,  $\delta_{\{\beta>1\}}(R_D, \lambda^{(p)})$  is bounded and the secondary network can achieve a throughput scaling of  $\Theta(\sqrt{\lambda^{(s)}/\log(\lambda^{(s)})})$  given  $q^{(s)} =$

$O(1/\log(\lambda^{(s)}))$ , regardless of the value for  $R_D$ . In addition, the relative secondary network AMG is derived as

$$\frac{\chi_{\{\beta>1\}}^{(s)}}{\chi} = \lim_{\lambda^{(p)} \rightarrow \infty} \delta_{\{\beta>1\}}(R_D, \lambda^{(p)}) = e^{-(R_I^{(ps)}/R_I^{(p)})^2},$$

for  $R_D \leq R_I^{(ps)} - R_I^{(s)}$ ; otherwise

$$\begin{aligned} e^{-a_2} &\leq \frac{\chi_{\{\beta>1\}}^{(s)}}{\chi} \leq \left[ e^{-\left(\frac{R_I^{(ps)}}{R_I^{(p)}}\right)^2} + e^{-\frac{((R_I^{(ps)})^2 - R_D^2)^+}{(R_I^{(p)})^2}} \right] \\ &\quad \cdot e^{1-2a_2} \left( \frac{1 - \exp\left(-\frac{\lambda^{(s)}(1-q^{(s)})\pi(R_r^{(s)})^2}{2}\right)}{1 - \exp\left(-\frac{\lambda^{(s)}(1-q^{(s)})\pi(R_I^{(s)})^2}{2}\right)} \right), \end{aligned}$$

where  $a_2 = (R_D/R_I^{(p)})^2$  and  $q^{(p)} = O(1/\log(\lambda^{(p)}))$ .

**Remark 4.** Note that even when requiring that the primary network AMG remains intact asymptotically (by setting  $R_D = R_r^{(p)}$ ), the secondary network can still achieve a non-trivial AMG greater than or equal to  $\chi \exp(-1/(1+l^{(p)}))$ . Interestingly, this bound increases as the ratio between the primary interference range and the transmission range increases. Furthermore, if the primary network allows a linear reduction in its AMG, the secondary network AMG can improve at least exponentially given that the secondary receivers are robust enough against primary interference (i.e.,  $R_I^{(ps)} \leq \alpha R_r^{(p)}$  with  $\alpha < 1$ ).

Next, we determine the secondary network throughput scaling and AMG when  $\beta < 1$ . First we introduce an upper bound for the probability of successful transmission in the secondary network. Using Lemmas 2 and 3, the derivation replicates the procedure in (13) except that

$$\begin{aligned} \Pr(\Lambda_{3,X_{n+1}^{(s)}} \mathbf{1}_{\beta>1} \mid \tilde{\Lambda}_{1,X_n^{(s)}}) &= \int_{r=0}^{R_r^{(s)}} e^{-\lambda^{(p)} q^{(p)} (|B_{R_I^{(ps)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(s)})^2} \\ &\leq e^{-\lambda^{(p)} q^{(p)} (|B_{R_I^{(ps)}}(0,0) - B_{R_D}(r,0)|)} \left( \frac{R_I^{(ps)} + R_D}{R_r^{(s)}} \right)^2 \\ &\quad + \int_{r=R_I^{(ps)} + R_D}^{R_r^{(s)}} e^{-\lambda^{(p)} q^{(p)} (|B_{R_I^{(ps)}}(0,0) - B_{R_D}(r,0)|)} \frac{2rdr}{(R_r^{(s)})^2} \\ &\leq \left( \frac{R_I^{(ps)} + R_D}{R_r^{(s)}} \right)^2 + e^{-\lambda^{(p)} q^{(p)} \pi (R_I^{(ps)})^2} \left( 1 - \left( \frac{R_I^{(ps)} + R_D}{R_r^{(s)}} \right)^2 \right). \end{aligned}$$

Observe that, based on this upper bound the secondary network throughput is asymptotically zero when  $R_D = \omega(R_r^{(p)})$ . Therefore, the secondary user's detection range should not exceed  $kR_r^{(p)}$  (for some positive constant  $k$ ) when the secondary network is sparser than the primary network. In this case we can obtain a new upper bound<sup>9</sup>, by setting  $\tilde{q}^{(s)} = (\lambda^{(s)} \pi (R_I^{(s)})^2)^{-1}$ , as  $\Pr(\Lambda^{(s)} \mathbf{1}_{\beta<1}) \leq$

<sup>9</sup>Simpler but looser.

$2 \exp\left(-\lambda^{(p)} q^{(p)} \pi \left(R_I^{(ps)}\right)^2\right) \Pr\left(\Lambda_{X_n^{(s)}}\right)$ . However, this is just an upper bound and the actual value could be significantly smaller. Hence, we find a lower bound for the probability of successful transmission in the secondary network when  $\beta < 1$  as follows:

$$\begin{aligned} \Pr\left(\Lambda^{(s)} \mathbf{1}_{\beta < 1}\right) &= \Pr\left(\Lambda_{3, X_{n+1}^{(s)}} \tilde{\Lambda}_{X_n^{(s)}} \mathbf{1}_{\beta < 1}\right) \\ &= \Pr\left(\tilde{\Lambda}_{X_n^{(s)}} \mathbf{1}_{\beta < 1} \mid \Lambda_{3, X_{n+1}^{(s)}}\right) \Pr\left(\Lambda_{3, X_{n+1}^{(s)}}\right) \\ &\geq \Pr\left(E_1 E_2 \mathbf{1}_{\beta < 1} \mid \Lambda_{3, X_{n+1}^{(s)}}\right) \Pr\left(\Lambda_{3, X_{n+1}^{(s)}}\right) \\ &\geq \tilde{q}^{(s)} \left(1 - \left(\frac{R_I^{(ps)} + R_D}{R_r^{(s)}}\right)^2\right) \\ &\quad \cdot e^{-\lambda^{(s)} q^{(s)} \pi \left(R_I^{(s)}\right)^2} e^{-\lambda^{(p)} q^{(p)} \pi \left(R_I^{(ps)}\right)^2} \\ &\geq \left(1 - \left(\frac{R_I^{(ps)} + R_D}{R_r^{(s)}}\right)^2\right) e^{-\lambda^{(p)} q^{(p)} \pi \left(R_I^{(ps)} + R_D\right)^2} \\ &\quad \cdot \Pr\left(\Lambda_{X_n^{(s)}}\right), \end{aligned}$$

where  $E_1$  is the event that  $X_n^{(s)}$  lies in  $B_{R_r^{(s)}}(X_{n+1}^{(s)}) - B_{R_I^{(ps)} + R_D}(X_{n+1}^{(s)})$  and initiates a transmission, and  $E_2$  is the event that none of the secondary users (except  $X_n^{(s)}$ ) in  $B_{R_I^{(s)}}(X_{n+1}^{(s)})$  initiate transmissions. Observe that this lower bound is strictly greater than zero given  $R_D = O\left(R_r^{(p)}\right)$  and  $R_I^{(ps)} \leq R_I^{(p)} = o\left(R_r^{(s)}\right)$ . Consequently, we establish that the secondary network can achieve a throughput scaling of  $\Theta\left(\sqrt{\lambda^{(s)}/\log(\lambda^{(s)})}\right)$  with the ALOHA access probability  $q^{(s)} = O(1/\log(\lambda^{(s)}))$  even when the secondary nodes are much more sparsely distributed in the network than the primary nodes. Moreover, the secondary network AMG is bounded as

$$e^{-\frac{(R_I^{(ps)} + R_D)^2}{(R_I^{(p)})^2}} \leq \frac{\chi_{\{\beta < 1\}}^{(s)}}{\chi} \leq 2e^{-\left(\frac{R_I^{(ps)}}{R_I^{(p)}}\right)^2}. \quad (14)$$

**Remark 5.** This result complies with the earlier conclusion that if  $R_D = \omega\left(R_r^{(p)}\right)$ , the secondary network throughput goes to zero asymptotically. Moreover, from (12) we have that setting the detection range as  $0 < R_D \leq k R_r^{(p)}$  (for any constant  $k$ ) will not improve the primary network AMG; therefore, it is favorable for the secondary network to blindly access the channel according to the traditional ALOHA medium access scheme, in which case the secondary AMG would be exactly equal to  $\exp\left(-\left(R_I^{(ps)}\right)^2 / \left(R_I^{(p)}\right)^2\right)$  that coincides with the lower bound in (14).

## V. CONCLUSION

We studied the interaction between two overlaid ad-hoc networks, one with legacy primary users who are licensed to access the spectrum and the other with cognitive secondary users who opportunistically access the spectrum. We showed that if the secondary network is denser than the primary

network, we can guarantee the same throughput scaling for both networks as that for a single network, as long as they deploy proper random access schemes. Furthermore, with the newly defined performance metric, the asymptotic multiplexing gain (AMG), we quantified how the asymptotic network performance is affected by the mutual interference between the two networks. In addition, for the first time we studied the throughput performance of an overlaid cognitive network in which secondary nodes are less densely distributed than the primary users and showed that even in this scenario, both networks can achieve the single-network throughput scaling. However, unlike the case of denser secondary network, spectrum sensing cannot improve the primary network AMG asymptotically while maintaining a nonzero throughput for the secondary network.

## APPENDIX A PROOF OF LEMMA 2

We first identify the event in which a secondary node refrains from transmission. Recall from Section IV-B, a secondary node initiates a transmission with probability  $q^{(s)}$  if it does not detect any primary transmitters in its detection range, denoted here by a disk  $\tilde{B}_{R_D}$  with radius  $R_D$ . In other words, a secondary user refrains from transmission (event  $I_2$ ), if there exists at least one primary transmitter in  $\tilde{B}_{R_D}$  (denoted by event  $I_{2,1}$ ) or it detects no primary transmitters but decides to defer transmission with probability  $(1 - q^{(s)})$  (denoted by event  $I_{2,2}$ ). Observe that events  $I_{2,1}$  and  $I_{2,2}$  are disjoint. We now determine the probability that there are no primary transmitters in a disk  $B_R$  of radius  $R$  (denoted by event  $I_1$ ) given that there is a single non-transmitting secondary user in the network (event  $I_2 = I_{2,1} \cup I_{2,2}$ ):

$$\begin{aligned} \Pr(I_1 \mid I_2) &= \frac{\Pr(I_1 \cap I_2)}{\Pr(I_2)} = \frac{\Pr(I_1 \cap I_{2,1}) + \Pr(I_1 \cap I_{2,2})}{\Pr(I_{2,1} \cup I_{2,2})} = \\ &= \frac{\Pr(I_1 \mid Q) \Pr(Q) + \Pr(I_1 \mid \bar{Q}) \Pr(\bar{Q}) + \Pr(I_1 \mid I_{2,2}) \Pr(I_{2,2})}{\Pr(I_{2,1} \cup I_{2,2})}, \end{aligned} \quad (15)$$

where  $Q$  is the event that none of the primary transmitters fall in  $B_R \cap \tilde{B}_{R_D}$ . It is immediate that  $\Pr(I_1 \mid \bar{Q}) = 0$  and

$$\begin{aligned} \Pr(I_1 \mid I_{2,2}) \Pr(I_{2,2}) &= \exp(-\lambda^{(p)} q^{(p)} |B_R - B_R \cap \tilde{B}_{R_D}|) \\ &\quad \cdot (1 - q^{(s)}) \exp(-\lambda^{(p)} q^{(p)} |\tilde{B}_{R_D}|) \\ &= (1 - q^{(s)}) \exp(-\lambda^{(p)} q^{(p)} |B_R \cup \tilde{B}_{R_D}|), \end{aligned} \quad (16a)$$

$$\begin{aligned} \Pr(I_1 \mid Q) \Pr(Q) &= \exp(-\lambda^{(p)} q^{(p)} |B_R - B_R \cap \tilde{B}_{R_D}|) \\ &\quad \cdot \exp(-\lambda^{(p)} q^{(p)} |B_R \cap \tilde{B}_{R_D}|) \\ &\quad \cdot \left(1 - \exp(-\lambda^{(p)} q^{(p)} |\tilde{B}_{R_D} - B_R \cap \tilde{B}_{R_D}|)\right) \\ &= \exp(-\lambda^{(p)} q^{(p)} |B_R|) - \exp(-\lambda^{(p)} q^{(p)} |B_R \cup \tilde{B}_{R_D}|). \end{aligned} \quad (16b)$$

Substituting (16a) and (16b) into (15) we have,

$$\begin{aligned}
& \Pr(I_1 | I_2) \\
&= \frac{e^{-\lambda^{(p)} q^{(p)} |B_R|} - q^{(s)} e^{-\lambda^{(p)} q^{(p)} |B_R \cup \tilde{B}_{R_D}|}}{1 - q^{(s)} e^{-\lambda^{(p)} q^{(p)} |\tilde{B}_{R_D}|}} \\
&= e^{-\lambda^{(p)} q^{(p)} |B_R|} \left( \frac{1 - q^{(s)} e^{-\lambda^{(p)} q^{(p)} |\tilde{B}_{R_D} - B_R \cup \tilde{B}_{R_D}|}}{1 - q^{(s)} e^{-\lambda^{(p)} q^{(p)} |\tilde{B}_{R_D}|}} \right) \\
&\leq e^{-\lambda^{(p)} q^{(p)} |B_R|} = \Pr(I_1).
\end{aligned}$$

Therefore, knowing that there exists a non-transmitting secondary user decreases the probability that there are no primary transmitters in a certain area.

#### APPENDIX B PROOF OF LEMMA 3

Let  $I_1$  denote the event that there are no primary transmitters in a disk  $B_R$  of radius  $R$ ,  $I_2$  denote the event that there exists a non-transmitting secondary user, and  $I_3$  denote the event that there is a transmitting secondary user. The existence of a transmitting secondary user implies that there are no primary transmitters present in its detection range (a disk  $\tilde{B}_{R_D}$  with radius  $R_D$  centered at the transmitting secondary user). We can upper-bound the probability that the area  $B_R$  is void of primary transmitters given there exist both a transmitting and a non-transmitting secondary users, as follows:

$$\begin{aligned}
\Pr(I_1 | I_2 \cup I_3) &= \frac{\Pr(I_1 \cap (I_2 \cup I_3))}{\Pr(I_2 \cup I_3)} \\
&\leq \frac{\Pr(I_1 \cap I_2) + \Pr(I_1 \cap I_3)}{\Pr(I_2 \cup I_3)} \\
&\leq \frac{\Pr(I_1 \cap I_2)}{\Pr(I_2)} + \frac{\Pr(I_1 \cap I_3)}{\Pr(I_3)} \\
&= \Pr(I_1 | I_2) + \Pr(I_1 | I_3) \\
&\leq \Pr(I_1) + \Pr(I_1 | I_3),
\end{aligned}$$

where the third line is due to the fact that  $\Pr(I_2 \cup I_3) \geq \max\{\Pr(I_2), \Pr(I_3)\}$  and the last inequality is a direct consequence of Lemma 2.

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